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An Introduction to Mathematics

A STUDY OF THE NATURE OF MATHEMATICAL KNOWLEDGE

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PRELIMINARY NOTE

For several years the author has had the privilege of giving lectures upon the nature of mathematics to students in the University of New Mexico. One set of lectures has been given to certain classes of beginning students in college mathematics as it is the belief of many mathematicians that little actual mathematical thinking can be indulged in without some comprehension of the meaning of mathematics. An additional advanced course of lectures of the same fundamental nature has been made available to advanced students in mathematics with the hope that they would appreciate many of the problems which face the creative mathematician. As a result of this experience it would seem that a short monograph of popular appeal upon the nature of mathematics might be welcomed. Of course, there are a few excellent books available to the student who cares to work seriously in the field. However, these books contain difficult vocabularies or symbolic languages which hide difficult concepts. So it is the purpose of this short work to present a simple introduction to modern mathematical thought, omitting completely many of the important extensions which are occupying the attention of mathematicians.

One must realize that the decision as to what is mathematics or what is not mathematics is, to a great extent, arbitrary. Hence not all mathematicians agree. Some mathematicians prefer to be called "pure mathematicians," and others prefer to be classified as "applied mathematicians." The purest of "pure mathematicians" refuses to soil his hands in the laboratory, and his meditations are the envy of modern philosophers to such an extent that philosophers are now studying mathematics. The extremist among the "applied mathematicians" goes to sleep while a theory is expounded; he wants a formula; and he wants to know when and how such a formula is used. If this presentation seems to be somewhat "pure," it is merely because the author deems such an approach to be the proper one, rather than that he lacks an appreciation of the viewpoint of the practical man. Naturally few of such men of practice are to be expected to understand or to appreciate such thoughts from the pen of their ally, the mathematician, as are recorded here. Moreover, the philosopher and the psychologist may be critically inclined toward the tendencies of modern mathematics. However, many sound critics are seeing in the new endeavors of mathematics a potent medium for integration within the sciences. Certainly modern mathematical philosophy has the whole-hearted support of some of the greatest thinkers of the age.

One frequently hears that mathematics is static. There could be no greater misconception. Without doubt there have been long periods of time when mathematics was virtually a static art. Creative possibilities were not recognized. But we are now living in the greatest age of all time from the standpoint of mathematical development. Many of the outstanding achievements in physical science have been due directly to the growth of mathematics. Perhaps this short presentation will indicate to the non-professional mathematician some of the opportunities for study in the field of mathematics. At least, he will recognize that the art has advanced considerably beyond the stage represented by the former high school requirement that theorems in geometry must be memorized.

It is impossible in such a work as this to give proper credit for all of the ideas expressed. Some may be original, but probably most are not. The author does acknowledge, however, his great indebtedness to Professor R. L. Wilder of the University of Michigan. His masterful classroom presentation of such difficult topics as "Analysis Situs" and "Topology" has served to stimulate each one of his students to think more about the true nature of mathematical knowledge.

AN INTRODUCTION TO MATHEMATICS

At every party which a mathematician attends the final scores of the participants in the games are placed before him for totaling. This may prove to be very embarrassing as the mathematician may have had little opportunity to add great strings of numbers previously. This, however, typifies the attitude of most people toward the mathematician. A mathematician to them is a computer. The truth of the matter is that many internationally known mathematicians have less opportunity to manipulate numbers than the ordinary housewife as she figures upon her grocery budget. Many mathematicians are puzzled over the elementary processes of accounting, and stories are told of famous mathematicians who have difficulty in the making of change. The physicist, as the computer, may have various needs for mathematics in his work; but he may be a very poor mathematician. A similar statement may be made in regard to the geometer or the surveyor. It must be understood at once that mathematics is something much more fundamental than its applications. The dictionary encourages this popular misconception when it defines mathematics as the science related to number and space.

Bertrand Russell has defined mathematics as the "class of all propositions of the type 'p implies q.' "[1] It seems desirable in this study to define a mathematical system as the resultant of two components, namely, a set of axioms and a system of logic. Mathematics, then, will be defined as the totality of such mathematical systems. Although this is essentially the same as the definition promulgated by Russell, it may prove somewhat more acceptable here. The construction involved as the two components mentioned above are combined is frequently characterized as the mathematical method of analysis.

The mathematical method is becoming increasingly important in science. Certainly no better statement of the

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1. Russell, "Principles of Mathematics," vol. 1, p. 1.

method and its importance has appeared than that which is due to G. C. Evans, outstanding mathematical economist. He said, "The systematization which occurs in a theoretical science, as we may properly call it in order to distinguish it from a natural or an applied science, is a process which is apt to come late in the development of a subject. Evidently some fields of knowledge are hardly ready for it, for it is typified by a free spirit of making hypothesis and definitions rather than a mere recognition of facts. But when we find this feeling for hypothesis and definition and, in addition, become involved in chains of deductive reasoning, we are driven to a characteristic method of construction and analysis which we may call the mathematical method. It is not a question as to whether mathematics is desirable or not in such a subject. We are, in fact, forced to adopt the mathematical method as a condition of further progress."[2]

In a recent article by Lapicque one finds some most interesting remarks pertaining to the use of the mathematical method in physiology. He wrote, "Formerly, not very far back in the history of humanity, let us say a century ago, almost everything was unknown concerning the physiology in the labyrinth of the living body. Magendie said: 'I wander around there like a rag picker, and at each step I find something interesting to put in my basket.' This maxim horrified my teacher, Dastre, who was wont to say: 'When one doesn't know what he is looking for, he doesn't know what he finds.' For him the ideal of physiological research would have been to conceive in the quiet of one's study a theory explaining such and such a phenomenon, known but not understood (physiology is full of phenomena of this character), then to find, still by meditation, the experiment capable by a yes or a no, of proving or disproving the theory. One would come then some morning to the laboratory, and that very evening the matter would be decided. These two tendencies, each in its amusingly exaggerated form, seem to me to serve the purpose of characterizing the temperament of naturalists and that of physicists. In proportion as physiology develops, the discoveries for ragpickers become more rare, and the possibility of working as Dastre dreamed is approaching."[3]

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[2] Evans, "Mathematical Introduction to Economics," p. 113.

In postulational technique, as indicated in each of the two previous quotations, a mathematical system has its beginning. To indicate the necessary emphasis upon the postulates the mathematical method is frequently known as the postulational or the axiomatic method. Hence, it seems appropriate to consider at once the first component involved in a mathematical system, namely, the axioms. In this connection it may be observed that the modern mathematician no longer distinguishes between postulates and axioms. Such words as axiom, postulate, and assumption are usually used synonymously.

Probably Aristotle (384-322 B. C.) was the first person to give serious consideration to the nature of axioms. It must be admitted, however, that his study was probably influenced tremendously by the previous work of Plato. The logician looks to Aristotle as the father of the science of logic. Aristotle's original system of logic was undoubtedly framed upon models taken from mathematics; but the mathematician has been inclined to ignore the important work of Aristotle upon the study of axioms and his other contributions to deductive science. In 1904, Heiberg made a most valuable collection of mathematical extracts from Aristotle.[4] These quotations indicate that Aristotle had a rather modern view of the fundamental nature of mathematical knowledge. Yet a recent mathematical historian has completely omitted the name of Aristotle from his treatise upon the history of mathematics.

One passage from Aristotle's "Posterior Analytics" seems sufficiently illuminating that rather extensive quotations from it are here given. After asserting that every demonstrative science starts from necessary principles, he proceeds:

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[3] L. Lapicque, "L'orientation actuelle de la Physiologie," in "L'orientaton actuelle des sciences," Paris, 1930.

[4] "Mathematisches zu Aristoteles" in "Abhandlungen zur Gesch. d. math. Wissenschaften," Heft, pp. 1-49.

> By first principles in each genus I mean those the truth of which it is not possible to prove. What is denoted by the first (terms) and those derived from them is assumed; but, as regards their existence, this must be assumed for the principles but proved for the rest. Thus what a unit is, what the straight (line) is, or what a triangle is (must be assumed) : and the existence of the unit of magnitude must also be assumed, but the rest must be proved. Now of the premises used in demonstrative sciences some are peculiar to each science and others common (to all), the latter being common by analogy, for of course they are actually useful in so far as they are applied to the subject-matter included under the particular science. Instances of the first principles peculiar to a science are the assumptions that a line is of such and such a character, and similarly for the straight (line) : whereas it is a common principle, for instance, that, if equals be subtracted from equals, the remainders are equal. But it is enough that each of the common principles is true so far as regards the particular genus (subject-matter) ; for, (in geometry) the effect will be the same even if the common principle be assumed to be true, not of everything, but only of magnitudes, and, in arithmetic, of numbers.
>
> Now the things peculiar to the science, the existence of which must be assumed, are the things with reference to which the science investigates the essential attributes, e. g., arithmetic with reference to units, and geometry with reference to points and lines. With these things it is assumed that they exist and that they are of such and such a nature . . . For every demonstrative science has to do with three things, (1) the things which are assumed to exist, namely the genus (subject-matter)in each case, the essential properties of which the science investigates, (2) the common axioms so-called, which are the primary source of demonstration, and (3) the properties with regard to which all that is assumed is the meaning of the respective terms used.

The most famous of all mathematical systems was organized by Euclid shortly after the time of Aristotle. Euclid was the first professor of mathematics in the famous university of Alexandria. Hence, his period of activity may be dated approximately at 300 B. C. Virtually nothing is known of Euclid. Probably he was a Greek rather than an Egyptian. Certainly, however, he may be characterized as the first great mathematical organizer. His monumental work, "The Elements," in thirteen books, is known to every school boy. Practically all of the geometrical material in modern plane and solid geometry texts is contained in parts of six of the books (1, 3, 4, 6, 11, and 12) of "The Elements." How much of this material was original with Euclid remains a question. Undoubtedly he had before him the already famous geometrical text-book written by Theudius of Magnesia. Moreover, he probably was influenced in his scheme of organization by the ideas of various pupils of Aristotle and of Plato. Certainly his organization with an explicit statement of his definitions and of his axioms at the very start of his analysis was a landmark in the history of mathematical thought. This is true in spite of the fact that the definitions and axioms would not meet modern requirements of rigor.

Following the statement of the twenty-three definitions which occur at the start of Book 1 of "The Elements" are found the axioms. These axioms are divided into two classifications, the postulates and the "common notions." The common notions are those assumptions which Aristotle would say are common to all sciences and which would today be described as axioms of logic. Typical of the set of five common notions is the first one which states, "Things which are equal to the same things are also equal to one another." The five postulates are axioms peculiar to the particular study in which Euclid was interested. By far the most noteworthy of the postulates is the fifth one which is usually known as the parallel postulate. It is, "Let the following be postulated that, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles." It is somewhat doubtful just what viewpoint Euclid was disposed to take in regard to his axioms. However, T. L. Heath, who has been characterized as the greatest modern authority upon the history of mathematics of the Greek period, says that we might imagine Euclid as saying, "Besides the common notions there are a few other things which I must assume without proof, but which differ from the common notions in that they are not self-evident. The learner may or may not be disposed to agree to them, but he must accept them at the outset on the superior authority of his teacher, and must be left to convince himself of their truth in the course of the investigation which follows."[5]

D. E. Smith says that more than a thousand editions of Euclid's "Elements" have appeared since the first one printed in 1482, and for 1800 years before that, manuscript copies dominated all teaching of geometry. In fact, most mathematicians think of Euclid as a document rather than as a man. It was not until the close of the eighteenth century that mathematicians considered making any essential change in the geometry of Euclid. The virtual sanctity with which the geometry was held is one of the amazing things in mathematical history. Mathematicians apparently found it impossible to conceive of any other geometry or of any other type of mathematical system with an axiomatic background as we now think of it. True, there was some dissatisfaction with the parallel postulate but even before the time of Euclid the notion of parallels seemed to cause concern and the topic furnished traditional controversy. It appears that there was no desire among mathematicians to actually cast out the postulate. Some indulged in the amusing task of attempting to prove it using the other axioms of the system. We can well imagine that others denounced the assumption for other reasons.

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[5] Heath, "The Thirteen Books of Euclid's Elements," vol. 1, p. 124.

The first part of the nineteenth century furnished the second great landmark in the history of mathematical thought. The era of Bolyai and Lobachevsky really marked the dawn of modern mathematical thinking. The invention of non-Euclidean geometry by these two men, independent of each other, marked the first turning away from the dominance of Euclid. A foundation for their work had been laid by Saccheri, Gauss, and others; but when, at the age of twenty-one, Bolyai wrote to his father saying, "I have created a new universe from nothing," one period had been finished and a new one had begun.

The only real difference between the geometry of Euclid and the geometry invented by Bolyai and Lobachevsky is that the parallel postulate of Euclid is replaced by another assumption. One might consider that the two inventors looked upon the matter as a sporting proposition to see what type of theorems would result if such a replacement were made. Certainly, as a result of the particular change which the two men both made, a strange geometry resulted. Those theorems of Euclid that were independent of postulate 5 still held in the new geometry. Other theorems in the old geometry were replaced by such amazing conclusions as this: that in a plane, instead of one line, two lines could be drawn through a point parallel to a given line and that through this point an infinite number of lines might be drawn lying in the angle between the first two and such that no one of the infinity of lines would intersect the given line.

Reimann demonstrated still another geometry in 1854 at Göttingen. Replacing the parallel postulate by still another assumption he obtained a geometry in which all lines were of finite length and the sum of the angles of a triangle was greater than two right angles.

To the novice such conclusions seem strange and probably ridiculous. To the mathematician the results were extremely important. The thought that the axioms underlying a mathematical system must be "obvious truths" slowly became a thing of the past, except in certain high school class rooms. It no longer is demanded that the postulates or the resulting theorems should appeal to a person's notion of truth. Some of the axioms may seem to be true, some may seem to be false, while others may be indeterminate in the sense that they do not possess the property of truth or falsity; yet the resulting system is accepted as a mathematical system if it is logically consistent. In fact, consistency, not truth, is the key word to mathematical thought. Although there are certain properties which a good set of axioms should possess (These will be discussed later), yet the axioms are to a great extent merely arbitrary starting statements.

Recently the Harvard University Press has started to publish ten volumes of the Collected Papers of Charles Sanders Peirce, famous Harvard Logician. Some of the statements of Peirce, as they appear in the early volumes, are so remarkably clear and concise upon the role which the axioms play in mathematical thought that they are quoted. Here is what he said:

> Every science has a mathematical part, a branch of work which the mathematician is called in to do. We say, "Here, mathematician, suppose such and such to be the case. Never you mind whether it is really so or not: but tell us, supposing it to be so, what will be the consequence?" (Vol. 1, p. 54.)
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> The business of the mathematician is to frame an arbitrary hypothesis, which must be perfectly distinct at the outset, so far, at least, as concerns those features of it upon which mathematical reasoning can turn, and then to deduce from this hypothesis such necessary consequences as can be drawn by diagrammatical reasoning. (Vol. 1, p. 241.)
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> Mathematics makes no external observations, nor asserts anything as a real fact. When the mathematician deals with facts, they become for him mere hypotheses; for with their truth he refuses to concern himself. (Vol. 3, p. 270.)

> I consider that the business of drawing demonstrative conclusions from assumed premises, in cases so difficult as to call for the services of a specialist, is the sole business of the mathematician. (Vol. 4, p. 10.)
>
> Mathematics is the study of what is true of hypothetical states of things. That is its essence and definition. (Vol. 4, p. 193.)

During the past century higher and higher standards have been required of postulational technique. The systematic study of the problem probably originated with Peano in 1889. In the United States important contributions were made by E. H. Moore and his followers, and by others during the early part of the twentieth century. However, the most influential work was the course of lectures which was given by Hilbert upon Euclidean geometry at the University of Göttingen during the winter semester of 1898-99, and which was subsequently published. Two editions of a translation by Townsend have appeared in this country. The greatest fault with the original work of Euclid was the fact that he frequently found it necessary to use an assumption which was not explicitly stated at the start of the work. Of course, a similar criticism might be made of the analysis of any high school boy as he proves his geometry theorems. Such matters, as well as other difficulties in the work of Euclid, were carefully regarded by Hilbert. The axioms, which were classified into five groups, were shown not to contradict one another. Also he made a study of the problem as to whether any one axiom might be known as a theorem and be demonstrated as a consequence of the remaining axioms. His study of this latter problem was rather remarkable in spite of the fact that E. H. Moore subsequently showed that one of the axioms was a logical consequence of the other axioms.

During recent years there has been a most interesting tendency to develop new mathematical systems. Virtually the only requirement for a new mathematical system is a new set of axioms; and, since the set of axioms is to great extent arbitrary, one aspect of recent mathematical industry is easily understood. Since it might be enlightening to the reader who is unfamiliar with this tendency in mathematics to observe such a typical study, the recent publication, "Foundation of Point Set Theory," by Moore,[8] is recommended.

Many features of modern mathematics which may seem strange to the novice may be understood if one appreciates the use to which a set of axioms may be put. Frequently the mathematician is asked, "How can there be so many different kinds of space?" or "How can there be more than three dimensions?" It is essential again to recall that the mathematician is not interested in reality as the term is usually understood. Hence, he can have any kind of space or any kind of dimensional theory by the simple process of setting up the proper set of axioms.

Naturally, it is difficult for the mathematician to limit his attention to mathematical systems of the traditional type. Moreover, the modern definition of mathematics does not require that mathematical systems be restricted to those involving such words as line, space, or number. The work of Bridgeman upon the foundation of physics is being observed with considerable interest. Incidentally, Bridgeman is one of the shrewdest critics of some of the mathematical systems. However, with the exception of a very few scientists the mathematicians form the only group to see much benefit to be derived from the consideration of postulational technique.

Mathematicians believe that in an inductive science there are always certain fundamental assumptions which determine the course of action in that science. These assumptions are axioms in the mathematical sense. In most instances no one who is informed would care to argue upon the truth or the falsity of them. The only justification for such assumptions is that of common agreement of the workers in the field. The axioms change from age to age and may be responsible for intense activity in one field of science or perhaps for a period of stagnation. Of course, there is no deductive science when there are no axioms to furnish a starting point. In either a deductive science or in an inductive science the explicit recognition of the underlying assumptions would seem to be essential to any progress.

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[8] American Mathematical Society Colloquium Publications, vol. 13.

As an illustration of the type of axioms used in a natural science the so-called law of cause and effect stands out. The recent work of Heisenberg has focused considerable attention to this axiom. Even though it has undoubtedly been effective in the advancement of science yet its continued use may prove to be a detriment to further advance. Several decades ago Peirce anticipated this when he wrote, "The first metaphysical axiom to go to the tomb of extinct creeds must be the proposition that every event in the universe is precisely determined by cause according to inviolable law." Science must be complimented upon the fact that consistency with this axiom has been demanded. If one event arbitrarily called an effect is observed without the related cause, one is immediately supplied. New words may be coined to hide the ignorance involved. The interesting thing is that such words can only be defined or explained in terms of the event designated as the effect. Gravity is such a word.

To one who can conceive of no science without the axiom of cause and effect it might be suggested that there are many alternate possibilities for axioms to replace or to supplement the disputed axiom. Many mathematicians are interested in an axiom which states that the connecting link between the cause and the effect is the link of probability. Hence, the present axiom would merely be a special instance of the new axiom in which the connecting link is 100% or unity. Under such a provision science would become more interested in those situations in which a certain event takes place as the result of a certain cause 85% of the time or perhaps 15% of the time. Undoubtedly, many branches of science would be stimulated inasmuch as they may now be restricted to situations involving the 100%

link. The present axiom of cause and effect requires a supplementary axiom having to do with the ultimate simplicity of the universe. It appears to many that an axiom recognizing ultimate complexity in the universe is the more desirable. Then such an axiom as the one above, involving a probability link becomes more imperative. As a second alternate to the present axiom one may recall the suggestion of Karl Pearson in "The Grammar of Science," wherein he wrote, "Whenever a sequence of perceptions D, E, F, G, is invariably preceded by the perception C, C is said to be a cause of D, E, F, G, which are then described as its effects. No phenomenon or stage in a sequence has only one cause, all antecedent stages are successive causes, and as science has no reason to infer a first cause, the succession of causes must be carried back to the limit of existing knowledge." As a supplementary axiom to the present one, a certain amount of spontaneity in nature might be permitted. Of course, the face of the dogmatic traditionalist would turn pale at this suggestion.

Mathematicians have been very much interested in the philosophical controversy between the idealists or subjectivists and the realists. Certainly the consideration of the connection between the mental experience of an individual and the possible external reality which may have given rise to it is responsible for many axioms which are in current use and which many people are using without due appreciation. One assumes axiomatically that the mental experiences of another do not vary from his own under corresponding conditions of stimulation. Yet, as J. S. Turner has expressed so ably, "If the fundamental existences should differ in different persons, no one would be aware of the fact; for no contradiction would arise in the exchange of ideas. Thus, if an object which gives me the sensation of redness, gives another person the sensation which I would call yellow, but which he calls red, we will not be aware that our sensations are different. Again, if the fundamental ideas and memory should change alike in the same person, he or she would not be aware of any inconsistency."[7] Hans Driesch has pointed out that the assumption of the existence of an external reality is an extremely useful hypothesis. However, certain further assumptions are needed pertaining to such a reality. He suggests, "These assumptions are the following: The real is, as it is technically called, 'rational,' that is, the most general concepts and propositions of an ordering nature which we apply to appearance (that is, to 'our world') are valid also for the real. For example, a property of the real which we call A is not, as being this property A, also something other than A; and two properties of the real, plus two further properties, are four real properties; and so forth."[8] Many similar axioms in common use but rarely explicitly stated might be cited. The few here given, however, may serve to point out the importance of axiomatic consideration.

Mathematics has even gone further in the consideration of certain fundamental aspects of axioms. As yet in this study no attention has been paid to the language of the axioms. Technically, of course, such a consideration should come first in any discussion such as this. Some parts of speech have not caused the mathematician any concern. For example, he looks upon articles and conjunctions as non-essentials. They have been introduced for rhetorical and literary nicety. Other parts of speech have given him great concern. Especially has the matter of definition of words caused many a mathematician to stay awake at night. The mathematician does not intend to indulge in the fallacy of the dictionary. A dictionary will define word, A, in terms of word B. Word, B, will be defined in terms of word, C, etc. Soon it may be observed that a word in the sequence will be defined in terms of word, A. Hence, ultimately word, A, is defined in terms of word, A. Such a process is justified, of course, for dictionary purposes since the hope exists that somewhere in the sequence of words a word will be found which possesses meaning to the user of the dictionary. Thus word, A, can be interpreted. Such vicious circle analysis is not desirable in mathematics. In fact, mathematicians have come to the conclusion that certain fundamental words in any science must be left undefined. This may seem unfortunate but apparently no other course is open to us. This is less of a difficulty in mathematics than in other fields of thought probably since ultimately meaning is subtracted from a pure mathematical system. It is considered desirable in any mathematical system to reduce the number of undefined words to a minimum. There has been much study of this particular point. For example, it has been found that in Euclidean geometry it is possible to reduce the number of undefined terms to three. In other words, one can define all other necessary terms by starting with the set of undefined words: point, line, and congruent. An alternate set of words which gives equal success is the group: point, between, and congruent. Of course, if certain fundamental words are undefined, it follows that other words, which are defined in terms of them, are likewise devoid of any fundamental meaning. It is not surprising, then, that Bertrand Russell said, "Mathematics is the subject in which we never know what we are talking about, nor whether what we say is true."

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[7] Turner, "Mathematical Logic," Collegiate Press, Ames, Iowa.

[8] Hans Driesch, "Man and the Universe," p. 52.

As implied above, however, the concept of definition does have its place in mathematics. When it seems to be desirable to extend the vocabulary beyond the primitive words, new words may be introduced. However, such new words most be defined in terms of the primitive words. The precise manner in which the new words are to be related to the undefined words through the medium of the definition is still a matter of controversy. The addition of new words to the vocabulary of a mathematical system is used in most interesting fashion. Whenever a system does not give much promise for further extension and the worker upon that system has exhausted the necessary research to enable him to keep in good standing with his profession, the insertion of a new word will give him a new thing to play with. It may be possible to prove a tremendous number of new theorems about the new word.

Not only is a consideration of the language of an axiom important but the form of such a primitive statement is likewise a matter of concern. By common agreement the form is that of a logical proposition, that is, a statement in which something is affirmed or denied of a subject. However, at once a difficulty is encountered. The statement, "Two points determine a line," is a rather common axiom in geometry. The statement appears to involve an affirmation. But it must be recalled that the words, point and line, are undefined words; hence, they are without meaning. The geometer may be picturing a very small dot for his point and a narrow mark for his line; thus he may nod in approval when the axiom is quoted. Another person may think of a pin point for his point and a clothes line for his line; thus he may think the statement sounds very ridiculous. Either person has a right to his own interpretation of the words involved. The fact is that such a statement as the axiom quoted above may be either true or false according to the content given to the undefined words. As it stands, however, since the words do not possess content, we can not say that it is either true or false. So the statement is not an actual proposition; rather it is merely in the form of a proposition. The novice in mathematics may be considerably disturbed by the lack of content in his words. Many words used by mathematicians already possess some sort of a meaning to many people. For such people it probably is desirable to replace the words by symbols of a new type. For example, a common axiom might be written, "X is a Y." Certainly that sentence is in the form of a proposition but a person is not tempted to ascribe to it any properties of truth or falsity.

As the mathematician observes the statement, "X is a Y," he is at once reminded of the concept of function in mathematics. Although the function theory expert would object, it may be stated that a variable is a function of one or more other variables if the first or dependent variable depends upon the other or independent variables in such a way that when the independent variables are given, the dependent variable may be determined.

It may be observed, therefore, that an axiom written in the accepted form has the form of a proposition and the characteristics of a mathematical function. Hence, an axiom is said to be a typical propositional function. Frege used propositional functions in his work, but Bertrand Russell receives the credit for the real development of the idea. It might be interesting to read what Russell had to say in regard to the concept. He defined a propositional function thus: "A propositional function is simply an expression containing an undetermined constituent, or several undetermined constituents, and becoming a proposition as soon as the undetermined constituents are determined. If I say 'x is a man' or 'n is a number' that is a propositional function."[9]

It may be remarked, at this time, that the axioms are not the only propositional functions in a mathematical system. The statements, which are deduced logically from the axioms, are likewise, of necessity, propositional functions.

Mathematicians believe that the notion of propositional function is a very important one. It is almost strange that the thought is so recent. Perhaps many of the so-called propositions are in reality propositional functions. C. J. Keyser has expressed an interesting thought in the statement: "That fact goes far to account for the endless disputations of men. For what can be a more prolific source of wasteful controversy than propositional functions uttered as propositions and regarded by mankind as propositions and thus believed to be true or false though in fact they are neither true or false?"[10]

So far the discussion has had to do with individual axioms and their properties. Mathematics has gone still further and has set up a group of requirements pertaining to the whole set of axioms which form the background of a mathematical system. These requirements are consistency, independence, and categoricalness. If the reader cares to go beyond the present discussion, a very fine presentation will be found in lecture 5 of Young's "Fundamental Concepts of Algebra and Geometry."

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[9] Russell, "Philosophy of Logical Atomism," in "The Monist," 1919.

[10] Keyser, "Mathematics and the Question of the Cosmic Mind," p. 7.

The notion of consistency in a set of mathematical axioms is simple if one is willing to treat the matter superficially. If, however, an individual desires to study the question seriously, he will find that there is no problem more abstract or difficult. Moreover, it must be understood that some of the greatest of modern mathematical thinkers are at present delving deeply into this topic.

A set of axioms may be spoken of as being consistent if it is impossible to deduce contradictory theorems from them. Of course, if contradictory theorems are obtained as a logical consequence of the axioms, the indication is that there exists contradiction within the axioms themselves. This concept would not be particularly difficult if it were not for the fact that mathematics makes use of a certain amount of logical terminology which probably needs some restrictions. This fact is appreciated acutely by the mathematician as he studies some of the modern logical paradoxes. A typical one might be interesting.

Probably one of the simplest as well as one of the best known of the logical paradoxes is the Russell paradox. It involves the word, class, which is one of the primitive terms of logic. However, it may be considered to be a word synonymous with such words as group or aggregate. This must not be thought of as a definition, however. The paradox, then, has to do with what will be known as an ordinary class. Any class which does not contain itself as an element is said to be such a class. As an illustration of this it may be said that the word, French, represents an ordinary class since the class of all words denoted by "French" does not contain the word, French. The word, English, is not ordinary, since the word, English, is contained in the class of words known as "English." The problem, then, is to consider the class, C, of all ordinary classes. Is such a class an ordinary class or a not-ordinary class? One will observe with a little thought that the assumption of either possibility leads to contradiction. It is impossible to tell, in other words, whether class C is ordinary or not-ordinary. Such a state of affairs is very sad indeed and individual mathematicians and logicians do not agree upon the remedy. The problem does imply the mathematical notion of infinity. Perhaps such a thought should be restricted or perhaps the word, class, needs some limitations. Suffice it to say, the idea of contradiction is not as simple as it may first seem to be.

Hilbert and his followers have been working for several years in an attempt to prove the whole of pure mathematics consistent. The effort involves the juggling of tremendous symbolic systems and many feel that there is little probability of success. Certainly they must be discouraged and mathematicians in general are disturbed by the recent finding of K. Gödel of Vienna. Gödel lectured two years ago at the distinguished American institution, the Institute for Advanced Study, upon the Princeton campus. His lectures there were impressive in their painstaking accuracy and thoughtfulness. In fact, as yet no refutation of his findings has been brought forward. He seems to have proved that a definite contradiction can be deduced from any proof which professes to prove the impossibility of the occurrence of contradiction in mathematics; that is, any attempt to demonstrate consistency in mathematics will lead in itself to inconsistency. This result is important not only for mathematics but for the whole of science.

Even if it be admitted that there exists no completely rigorous test for the consistency of a set of axioms, yet any mathematician desires to indulge in some form of check for the set which he may have created. The common method is to give a concrete representation of the system, in which the undefined terms are given content, and for which the axioms are all satisfied. If, then, the mathematician is able to picture a real or ideal situation (Is there any difference?) in which all of the axioms which now have value are represented in non-contradictory fashion, he reasons that there is no contradiction involved. This will be seen to be true if one considers that if any contradiction might arise in the original mathematical system a contradiction which corresponds to it would be observed in the material or ideal representation. Such considerations are not easy and it is small wonder that mathematicians are competing closely with the philosophers for the title of "absent minded professor." It must be observed that inability to discover a concrete representation in which all of the axioms are satisfied does not indicate that the set of axioms under consideration is inconsistent. It may only indicate lack of ability or good fortune on the part of the investigator. However, if such a representation is found, it follows that the set is consistent.

These ideas may be better understood if an actual set of axioms be tested for consistencey. A very simple set of axioms is the set employed in setting up simple order. The axioms are as follows:

(1) If a and b are not the same elements of C, then either a is less than b or b is less than a.

(2) If a is less than b, then a and b are not the same elements of C.

(3) If a is less than b, and b is less than c, then a is less than c.

In this set of axioms there appear several undefined terms. They are the letters, a, b, c, and C, and the expression, "is less than." Of course, the expression, "are not the same," may appear to have the same property, but it merely indicates lack of identity. The notion of identity belongs to the logicians, and mathematicians carefully avoid intrusion upon their domain. Mathematicians are supposed to have some intuitive notion of such concepts as the result of their contact, or lack of contact, with the logicians. For the necessary concrete representation of this system let a long single file of people be the representation of C. The small letters may be used to denote individual members of the file. The expression, "is less than," may be interpreted as "is standing somewhere in front of." It will then be seen at once that such a representation satisfies all of the axioms. As stated previously this fact demonstrates the consistency of the set.

The notion of independence as applied to a set of axioms is somewhat simpler than the concept of consistency. A set of axioms is said to be independent if no one of them is logically deducible from the others. Sometimes the property of independence is thought of as being an aesthetic property, for a mathematician is much more pleased with a set of axioms if he knows that the number of them has been reduced to an absolute minimum. This would not be true as long as one of the axioms might be proved as the logical consequence of the others and could thus be classified as a theorem instead of an axiom. However, it must be distinctly understood that independence is not an absolutely necessary attribute of a set of axioms. R. L. Wilder showed that one of Moore's original axioms, which served as the foundation for point set theory, could be deduced from the others, but this fact did not in the slightest destroy the work.

A satisfactory check for independence involves again the finding of a material or ideal representation for the axioms. However, this time it is desired to find a representation for which all of the axioms are satisfied except one. If such a representation can be found it serves to demonstrate that the one axiom can not be obtained as a consequence of the others. Again an illustration may serve to make this argument somewhat clearer. If in the set of axioms given previously for simple order, C is interpreted as the collection of all human beings from the start of time, the small letters are used to denote individual people, and the expression, "is less than," is interpreted to mean "is an ancestor of," then it will be readily observed that axioms (2) and (3) are true of the representation but axiom (1) is not. Consequently axiom (1) cannot be deduced from the other two, for if it could, it would likewise hold true for all situations for which the last two axioms are satisfied and one case has just been observed when this is not true. With a little thought, representations can be decided upon which will demonstrate the independence of the other two axioms. Of course, in more complicated sets of axioms the difficulty of applying the principle just considered is very great. Usually the investigator is forced to an ideal representation instead of a material one and this is very unfortunate. In any ideal picture the "intuitive" plays a very important role. Of course, as indicated previously, the intuitive may play a much more important role in the so-called material picture than some are willing to admit. This thought alone would be sufficient to justify the great interest of modern mathematicians in the ideal or subjective philosophy. For an extensive treatment of such matters the reader is referred to the recent work, "Mind and Nature," by Hermann Weyl, who is influential in the mathematical work of the Institute for Advanced Study.

The concept of categoricalness is much more difficult to discuss than the previous two properties of a good set of axioms. The mathematical concept of category is closely allied to the logical concept and one who is familiar with the principle in logic will have no difficulty understanding the similar concept in mathematics. It is admitted that the novice will have some hardship understanding any such superficial discussion as the following one.

Wilder is inclined to introduce the discussion of categoricalness with a consideration of what may be termed the completeness of a set of axioms. This notion of completeness resembles fundamentally the accepted notion of categoricalness and the two concepts may be identical. In fact, completeness in the sense in which it is used here is usually replaced by the word, categoricalness. Wilder then says, "A set of axioms is said to be categorical when all questions phrased in terms of the undefined terms can be answered on the basis of the set. Thus, a categorical set of axioms for plane geometry should be complete in the sense that on their basis all questions about plane geometry can be answered." Now there are no questions in plane geometry, as the term is understood here, which have to do with such properties as color. Hence, it is not demanded of the axioms that they should include what would be necessary to answer such questions.

The generally accepted definition of categoricalness is based upon the very interesting concept of isomorphism. It is necessary, therefore, that isomorphism should be discussed first. Any collection of things which satisfies a given set of axioms after the proper particularization of the undefined terms is known as a universe. Suppose, then, that two universes exist which satisfy the same set of axioms. Suppose, moreover, that there exists a correspondence between the individual members of the two universes of such a nature that any statement which is true of the first universe is likewise true of the second when the individual members involved in the statement are replaced by the corresponding members of the second universe. As a further supposition let the converse of this latter statement be true. Then the two universes are said to be isomorphic with respect to the set of axioms. Of course, there may be a very large number of universes isomorphic with respect to the same set of axioms. These universes, in a sense, may be seen to be essentially the same universe, although the language used in discussing them may be superficially different.

It may be said, now, that a set of axioms is categorical when every two universes which satisfy it are isomorphic. All of such universes may be said to be in the same category. The test for categoricalness will not be discussed here in any detail. Suffice it to say that the problem involves, of course, the establishment of the isomorphism of any two universes which satisfy the axioms. This is accomplished through the medium of a material representation which satisfies the axioms and which is then used in the sense of an intermediary to establish the isomorphism of any other two universes.

Of course, if the universes were considered by pairs, the task becomes hopeless. Hence, the common procedure involves the consideration of general universes.

It is beyond the scope of this brief study to consider the axiomatic background of particular mathematical systems. Perhaps, however, what has gone before will give the reader some insight into the careful study which has been indulged in by mathematicians upon sets of axioms. Those sets, which form the basis of such well known mathematical systems as arithmetic, algebra, and geometry, have furnished the background for much mathematical study and controversy. The unfortunate feature of the matter is that students even in the secondary grades are not given more opportunity to see the actual nature of mathematical thought and a little of the axiomatic background of the particular mathematical subject which they may be studying. It must be admitted that such considerations for immature students must be placed upon an elementary level and the presentation must be decidedly non-rigorous. These facts do not excuse the tendency of many teachers to completely ignore axiomatic technique.

As for the axiomatic method in general no finer summary could be found than that statement attributed to one of the characters in "South Wind" by Douglas, namely, "The older I get, the more I realize that everything depends on what a man postulates. The rest is plain sailing."

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The second component involved in the making of a mathematical system is logic. It is not the intention here to give a detailed description of the principles of that ancient field of study. However, there are some aspects of logic so closely tied up with mathematical considerations that they must be mentioned in even such a brief work as this.

It was the sixth century B. C. when Pythagoras saw the "human necessity for a clear conception of proof on which all sane mortals could agree." It was Aristotle, however, who first pronounced certain laws which he conceived as the background of rigorous thought. His laws, as usually stated at the present time, are:

(1) A is A. (The law of identity.)
(2) Everything is either A or not-A. (The law of excluded middle).
(3) Nothing is both A and not-A. (The law of contradiction.)

The last two of the above laws may be somewhat difficult of comprehension. By means of an interpretative device, used by most mathematicians and by many logicians, they become comparatively easy. Instead of thinking in terms of an abstract concept such as a property which several individual objects might possess, it is common for the mathematician to think in terms of the class of individuals which possess the property. For instance, if one should speak of the color, red, to a mathematician, he would probably not think of the abstraction which is denoted by red, rather he would think of the class of those objects which are red. This is approximately what is meant when it is said that the mathematician thinks of logical relations in extension rather than in intension. Moreover, the mathematician, as the logician, considers arbitrarily the product of two classes to be the new class composed of those members which are common to the two classes. Also the sum of two classes is taken as the new class composed of those members which belong to one or the other or both of the two given classes. So, then, if the minus sign be used to denote negation, the symbol 1 be used to denote the class of all objects (everything), and the symbol 0 be used to indicate the null class (nothing), the last two laws of Aristotelian logic become:

(2) $a + -a = 1$. (The law of excluded middle).
(3) $a \times -a = 0$. (The law of contradiction.)

Traditional logic is concerned almost exclusively with relations between terms. The problem is the drawing of a conclusion as to what must be true or cannot be true of such relations. The laws just given along with the rules of syllogism are the available means for forming a decision.

Beyond this simple statement it must be realized that a vast system of logic has grown up, based upon laws which have to do with certain ideas which are taken as fundamental. Such words or concepts as class, all, negation, "either-or," etc., are jealously guarded by the logician as his own property. The basic laws, involving these ideas, are universals in the sense that they do not belong in their application to any one science or subject. At times it is rather difficult for the mathematician to know just why the logician should have been given certain parts of the vocabulary to protect and certain laws to hold in his care when he feels that he could have used them to better advantage. Mathematicians along with other scientists have coveted some portions of sciences which by tradition belong to some one else.

A new appreciation of the fact that the primitive words of logic are undefined words and that the so-called fundamental laws of logic are axioms in the mathematical sense has been observed with interest by mathematicians. In fact, the structure known as logic, built upon an axiomatic foundation, certainly resembles any mathematical structure. Boolean Algebra is becoming increasingly popular and such a system of logic built with the use of a symbolic language and erected upon an axiomatic basis is readily grasped by any mathematician. Incidentally, it may be remarked for the benefit of the uninitiated that Boolean Algebra is definitely Aristotelian in character and the last two laws of Aristotelian logic as stated in symbolic form are typical conclusions.

If, then, the so-called laws, which form the basis for traditional logic, are mere axioms, it is not surprising that there is a tendency to modify them or perhaps replace them altogether under various circumstances. Perhaps the time is rapidly approaching when the type of logic employed will be determined by the nature of the problem to be studied, just as the mathematician now debates what kind of mathematics is adapted to the problem under consideration. Korzybski has been disturbed about the law of identity and he apparently feels that its universal use is unwarranted. Brouwer has, likewise, challenged the universal use of the law of the excluded middle. Certainly it must be admitted that traditional logic is founded upon a philosophy of a determinate universe and it is very doubtful whether that kind of a doctrine is everywhere applicable. There certainly are questions which can not be answered by any unrestricted "yes" or an unrestricted "no." Many of such questions are beyond human capability to answer. Should one say, "But even in such questions there is a proper answer"? Or should a place be reserved for indeterminacy in the universe and a logic be developed compatible with such a thought? It is very interesting that virtually all races of men are living under essentially the same system of logic. However, it has been detected in some of the aborigines that consistency is not an essential component of their logic.

The real stimulus to the study of the actual nature of logic as well as the possibility of having other kinds of logic in addition to the traditional system of Aristotle was furnished by the monumental work, "Principia Mathematica," by Whitehead and Russell. Few people have ever studied the three tremendous volumes which constitute this work. The language alone is enough to stop any one except the expert as the symbolic language originally developed by Peano is used. The point of immediate interest is the fact that the calculus of propositions developed in "Principia Mathematica" is definitely non-Aristotelian. The technical interpretations of implication and identity as employed in the work differ radically from the customary meanings assigned to them. Nevertheless, deductions are arrived at just as readily as through the use of traditional methods which have no application in the "Principia."

It is possible to develop the system of logic employed in the "Principia" by the use of a matrix method. For one who is accustomed to the use of tables the device will be simple to apply. If the truth or falsity of each of two propositions is known, the nature of a particular relationship between the two propositions may be determined by referring to a table which is termed the matrix. The nature of other relationships may then be determined by virtue of their connection to the relationship which is chosen as fundamental. By an extension of this scheme of action Lukasiewicz has constructed systems of logic in which propositions do not necessarily have one or the other of the two truth possibilities, "true" and "not-true," but can have any number of possibilities. Tarski has contributed to the further development of these systems.

Such developments have caused C. I. Lewis to issue the following declaration:

> (1) There are no "laws of logic" which can be attributed to the universe or to human reason in the traditional fashion. What are ordinarily called "laws of logic" are nothing but explicative or analytic statements of the meaning of certain concepts, such as truth and falsity, negation, "either-or," implication, consistency, etc., which are taken as basic.
>
> A "system of logic" is nothing more than a convenient collection of such concepts, together with the principles to which they give rise by analysis of their meaning.
>
> (2) There are an unlimited number of possible systems of logic, each such that every one of its laws is true and is applicable to deduction. These systems are alternatives in the sense that concepts and principles belonging to one cannot generally be introduced into another—because of fundamental differences of category."[11]

The significance of these results to mathematics is very great. Whereas it has been appreciated for some decades that the axiomatic component of mathematics is a variable one, now the task of the mathematician is considerably heightened by the realization that the logical component is also variable. No longer can the mathematician regard his logic as fixed and predetermined; rather he is faced by the fact that his very conclusions are relative to the type of logic employed as well as relative to the system of axioms involved. C. S. Peirce once said, "Logic studies the science of drawing conclusions, mathematics is the science which draws necessary conclusions." The implication of this statement was that mathematics had its task and logic had its particular work. However, the continued separation of the two fields is intolerable. Mathematicians must more and more be acquainted with the activities in logic. It is difficult for mathematicians to overcome the traditions of centuries and to feel an actual duty to develop a critical attitude toward logic, but it is necessary. Probably Brouwer's writings constituted such a bomb shell to mathematics partly because he urged that it was within the province of mathematics to subject logic to clinical treatment.

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[11] Lewis, "Alternative Systems of Logic," in "The Monist," 1932, p. 483.

But the last word in regard to the relation of mathematics and logic has not yet been said. Many students would not agree with the fundamental thesis of this discussion, namely, that mathematics is the resultant of the two components already discussed. Many consider that logic is nothing but a subdivision of a much vaster field called mathematics. Among the arguments advanced is the apparent fact that logical systems set up in symbolic languages resemble only very simple types of mathematical systems. On the other hand, there are many other students of the subject who contend very strongly that logic is an all-embracing subject and mathematics is merely a minor aspect of the field. This latter argument is sufficiently interesting to demand some special attention.

The particular school of mathematicians who profess to believe that mathematics can ultimately be reduced to logical concepts exclusively is known as the logistic school. By virtue of his remarkable work, "Formulaire," Peano is usually known as the forerunner of the school. As previously indicated members of the logistic school use a rather complex symbolic language such as first introduced by Peano. Also, among the pioneers, should be mentioned Frege, whose work was not appreciated until it was called attention to by Bertrand Russell. However, the culminating achievement of this group of men was the "Principia Mathematica," previously mentioned, by Russell and Whitehead. This tremendous work purports to demonstrate the tenet of the school, namely, that the whole of pure mathematics can be reduced to logic. It may be remarked that the treatise under discussion was written before the highly involved mathematical notion of transfinite number was introduced. Hence, there are now mathematical systems which are as yet unclaimed by logic. However, it is rather difficult to attack the contention of the authors of "Principia" in regard to such pure mathematical systems as arithmetic and its extensions. The method employed to obtain mathematics from purely logical concepts is comparatively easy to understand. As previously remarked, there are certain concepts which have been reserved for the play of the logician. Among such concepts one does not find the notion of number, zero, or the various mathematical operations as multiplication or addition. Only such harmless appearing concepts as class, negation, "either-or," etc., are allowed to the logician. Yet with amazing ingenuity the elementary mathematical concepts were defined in terms of the accepted logical concepts. It was then possible to use the work of Dedekind to go from the elementary concept of number (the integers) to the notion of irrationals. Ultimately all of the necessary laws of arithmetic were deduced. Certainly the work is a masterpiece of logical beauty. In recent years more and more objections have been raised to various devices employed in the "Principia" and the work is not regarded as highly now as at one time.

There are still others who have been observing the study of the nature of mathematical knowledge with a critical eye. The name of Brouwer has been already introduced into the discussion. He has warned that the symbolic structure usually identified with mathematics is merely the cloak of something much more fundamental in the way of a thought structure. He has indicated that when mathematicians manipulate the symbolic accompaniment of thought without due regard for the thought itself, there is trouble in store. He states very definitely that there is something which may be termed the fundamental "intuition" which must always serve as a guide to correct mathematical procedure.

The viewpoint of this study is essentially that of the formalist. The so-called formal school of thought acknowledges the leadership of David Hilbert. As indicated previously, Hilbert has encountered difficulty. Nevertheless, his ideas are essentially those of most American mathematicians. Mathematical systems are interesting to the formalist because of the structural properties involved. The individual symbols are devoid of meaning and they have no significance except as they are related to other symbols. Some mathematical systems may appear to involve the study of entirely different situations, yet there may be underlying structural properties which are similar. In other instances the dissimilarity of underlying structures may be the object of interest. This thought leads to the next short division of this work which has to do with the application of mathematical systems to the study of the physical or natural world in which we live.

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As has already been observed, a mathematical system is a skeleton structure having form but no content. Content can only be given to such a structure by the giving of meaning to the undefined terms involved. The task, then, of the applied mathematician is the seeking of a particular mathematical structure which possesses an axiomatic basis which will fit the particular application or situation under consideration if the undefined terms are given the proper meaning. It seems appropriate at times to consider a mathematical structure to be of the nature of a pattern which is available for superimposing upon a group of natural or physical phenomena if it fits. Naturally it would seem to be wise to have many patterns at hand and that is precisely the thought of the modern mathematician as he invents more and more mathematical structures. Perhaps many of the mathematical patterns will never be used but it is certainly true that many which are not now existent could be used. It may be argued by some that very few mathematical patterns are now actually in use by the so-called man of application. Of course, one must realize that few such men can use with facility any but the classical patterns. It may be argued, on the other hand, that the ability of many to use only a few systems may have caused a deficiency of scientific structures which are mathematically correlated.

The task of fitting a set of axioms to a natural situation is a delicate undertaking. In fact, it must be clearly understood that the ultimate fit is merely an apparent fit. It is impossible to cast the human element out of such considerations. Moreover, data furnished by a machine are trustworthy only within the limits of accuracy of the machine. Absolute accuracy is unattainable. Thus it may happen that several sets of axioms may appear to fit the same set of phenomena. As more observations are made and perhaps previous ones are corrected, some sets of axioms may be discarded as ill-fitting. At the same time new sets may appear to fit the revised information. At any stage in the study many sets of axioms will probably be found which seem to fit the situation under consideration equally well. Such a set of axioms and that which is deduced from them might then be considered to constitute a theory or perhaps a law. However, one must not make the mistake of considering such a pattern in the light of an explanation of what really takes place. The explanation is probably forever hidden from mortal eye and men are forced to consider only the superficial patterns which appear to fit. It is strange that so few men have appreciated this simple fact. Some of our outstanding thinkers of the day have pointed out this principle so clearly that quotations from them are in order.

Eddington says in "The Domain of Physical Science," an essay in "Science, Religion and Reality," "Leaving out all aesthetic, ethical, or spiritual aspects of our environment, we are faced with qualities such as massiveness, substantality, extension, duration, which are supposed to belong to the domain of physics. In a sense they do belong; but physics is not in a position to handle them directly. The essence of their nature is inscrutable; we may use mental pictures to aid calculations, but no image in the mind can be a replica of that which is not in the mind. And so in its actual procedure physics studies not these inscrutable qualities, but pointer-readings which we can observe. The readings, it is true, reflect the fluctuations of the world-qualities, but our exact knowledge is of the readings, not of the qualities. The former have as much resemblance to the latter as a telephone number has to a subscriber."

J. W. N. Sullivan said in a statement which is even more to the point: "It has become evident that, so far as the science of physics is concerned, we do not require to know the nature of the entities we discuss, but only their mathematical structure. And, in truth, that is all we do know. It is not realized that this is all the scientific knowledge we have, even of the familiar Newtonian entities."[12]

It is interesting to note that the elements of a physical situation which correspond to the axioms of a mathematical structure must be tied logically to the elements of the physical universe which corresponds to the upper reaches of the mathematical structure. Thus it is not unusual to find that through the medium of a mathematical system apparently unrelated events are correlated. Moreover, as the result of this same principle it is possible for mathematics to predict new physical entities. If the base of a mathematical system has a physical counterpart, but some portions of the structure have no corresponding physical entities, it is natural to seek the "missing elements." An actual indication can usually be obtained from a study of the mathematical structure as to what should be sought. There have been some spectacular discoveries as a result of mathematical prediction. Among these may be mentioned the discovery of the Hertzian wave, the discovery of the curvature of light rays as demanded by the Einstein theory, and the discovery of the outer planets of the solar system.

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[12] Atlantic Monthly, March 1933, p. 288.

There are dangers involved in the consideration of aspects of the physical universe through the medium of the related mathematical structure. One should especially beware of reading into the physical situation certain properties which may be inherent in the logical or mathematical structure itself rather than in its physical counterpart. It is suspected by some that such may be the case in the modern studies concerning the finiteness of the universe. Of course, there are other dangers, such as the improper interpretation in terms of physical entities of an abstract syllogism.

For the purpose of summary and for emphasis a recent quotation from the pen of G. A. Bliss, distinguished mathematician of the University of Chicago, is here given. He stated, "Mathematical theories have been of great service in many experimental sciences in correlating the results of observations and in predicting new data afterwards verified by observation. This has happened particularly in geometry, physics, and astronomy. But the relationship between a mathematical theory and the data which it is designed to relate is often misunderstood. When such a theory has been successful as a correlating agent the conviction is likely to become established that the theory has a unique relationship to nature as interpreted for us by the observations. Furthermore, it is sometimes inferred that nature behaves in precisely the way which the mathematics indicates. As a matter of fact, nature never does behave in this way, and there are always more mathematical theories than one whose results depart from a given set of data by less than the errors of observation." [13]

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[13] American Mathematical Monthly, October, 1933, p. 472.

Not only is the mathematician interested in the correct application of a mathematical system but he feels that he has a special obligation to evaluate mathematical or axiomatic thinking wherever it occurs. Correct mathematical thinking is such a rare thing that there seems to be some justification for more people studying mathematics. A few years ago a debate occurred between a physical scientist and a theologian. The mathematical fallacies of that debate were so striking as to justify a comment here. The physical scientist argued that after years of thought he was not able to deduce from any of his physical laws evidence for the existence of a Deity. The scientist, of course, did not understand the categorical property of a set of axioms. His years of thought were unnecessary, inasmuch as he had not placed a Deity in his axioms and most certainly, therefore, could not deduce such an existence from them. In the particular discussion the word, Deity, had not even been defined in terms of the undefined words which made the deduction quite impossible. The theologian countered by demonstrating from his axioms the existence of a Deity. He had taken the precaution to insert Deity in his axioms and, thus, was able to prove the desired proposition. The theologian was in advance of the scientist since he was using a categorical set of axioms. However, he did not understand that the so-called propositions of a mathematical system simply explain in greater detail what was really stated originally in the axioms. Proving a proposition does not create knowledge. Rather, every new proposition goes a little farther toward a clarification of the assumptions. To prove deductively the existence of a Deity from a set of assumptions simply indicates that the existence of a Deity was originally assumed but in a disguise not immediately recognized. Such a fundamental statement as "There is a God" should be made an axiom at once. Some would accept it and some would not. The question of the desirability of such an axiom would be determined by the consequences which arise from it.

Many of the conclusions of this work are debatable. Certainly some of the studies are abstract. But such is mathematics. There is nothing more fascinating than some of the modern considerations. Yet so few are acquainted with them. There is always the hope that one more will study such problems as are given here and will contribute to the advancement of the great art so necessary to future progress. As Braithwaite said in his preface to the remarkable group of essays by Ramsey upon the foundation of mathematics, "This is presented in the hope that it will stimulate others to think about the hardest things in the world."